INVISCID ROTATIONAL STAGNATION POINT FLOW (*)

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It is well known that the splitting streamline at a stagnation point on a wall in an inviscid rotational flow makes a finite angle with the normal to the wall. It is also well known that the stagnation streamline coming into a general stagnation point on a wall is normal to the wall in an irrotational flow. The purpose of this communication is to answer the question of what is the corresponding result with rotational flow and a general stagnation point.

We consider the flow to be steady and of constant density, on one side of a plane wall. An assumed form for the velocity field is chosen in which the vorticity component normal to the wall is zero. Since the vorticity component normal to a body in inviscid constant-density flow is generally zero there, this choice is not restrictive in investigating local behavior. The wall is the plane $z = 0$ in the cartesian space (ax, ay, az). The flow occupies the space $z \geqslant 0$. The quantity a is an arbitrarily chosen reference length. The normal flow approaching the wall is characterized by the reference velocity gradient V'.

The velocity is assumed to have the form

$$\mathbf{q} = {}^1/_2 V' a[F + x(H' - \alpha),\ G + y(H' + \alpha),\ -2H] \tag{1}$$

where F, G, H and α are functions of z alone, and primes denote differentiation with respect to z (except in V', of course). The vorticity corresponding to (1) is expressed

$$\nabla \times \mathbf{q} = {}^1/_2 V'[-G' - y(H'' + \alpha'),\ \ F' + x(H'' - \alpha'), 0] \tag{2}$$

Our method is to express the pressure gradient through the momentum equation, and then impose the condition that the curl of the pressure gradient be zero. We obtain the following equations

$$H\alpha' - H'\alpha = -\alpha_0, \qquad HH'' - {}^1/_2(H'^2 + \alpha^2) = -{}^1/_2(1 + \alpha_0{}^2) \tag{3}$$

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$$HF' - {}^1/_2(H' - \alpha)F = M = 0, \qquad HG' - {}^1/_2(H' + \alpha)G = N = 0 \tag{4}$$

with the right-hand sides of the equations initially arbitrary constants. The first two are evaluated through the boundary conditions

$$H(0) = 0 \qquad H'(0) = 1, \qquad \alpha(0) = \alpha_0 \tag{5}$$

which we impose. The parameter α_0 is basic in our investigation. With exceptions, the other two constants M and N can be set equal to zero by the following argument. Provided $\alpha_0 \neq 1$, if M is nonzero, a translation of the coordinate system in the x-direction can be found which makes M zero. We presume that such a translation has been made if necessary. Similarly, provided $\alpha_0 \neq -1$, we can set $N = 0$. In addition, inasmuch as we may interchange x and y without changing the form of the solution, we may restrict α_0 to nonnegative values without loss of generality.

Equations (3), with boundary conditions (5), are independent of equations (4). Since the right-hand sides of (3) were determined using (5), two additional boundary conditions are needed to determine the solution. This system and its solution will be termed primary.

With the primary solution known, Equations (4) for F and G are linear. We impose the boundary conditions

$$F(0) = 0, \qquad G(0) = 0 \tag{6}$$

here. This system and its solution will be termed secondary. The pressure distribution corresponding to a solution of these systems of equations is

$$p = p_{st} - {}^1/_2\rho V'^2 a^2({}^1/_4(1 - \alpha_0)^2 x^2 + {}^1/_4(1 + \alpha_0)^2 y^2 - Mx - Ny + H^2) \tag{7}$$

We turn now to the details of the solutions. It is convenient to consider various possible cases separately.

Axisymmetric case $\alpha_0 = 0$. The general primary solution is given by

$$\alpha = kH, \quad H = k^{-1}\sinh kz + H_0'' k^{-2}(\cosh kz - 1) \tag{8}$$

where $k = \alpha'(0)$. It is analytic. The general secondary solution is

$$F = f_0 \sqrt{H} \exp(-{}^1/_2 kz), \qquad G = g_0 \sqrt{H} \exp({}^1/_2 kz) \tag{9}$$

where f_0 and g_0 are arbitrary constants. Thus the secondary velocity behaves as $z^{\frac{1}{2}}$ near the wall, and the vorticity as $z^{-\frac{1}{2}}$. The stagnation streamline near the stagnation point has the shape described by

$$-\frac{x}{f_0} \doteq -\frac{y}{g_0} \doteq \frac{z^{1/2}}{2} \tag{10}$$

Nodal case $0 < \alpha_0 < 1$. In this case the primary streamlines on the wall form a nodal pattern. The only analytic primary solution is the special one $H = z$, $\alpha = \alpha_0$. The general primary solution has the form

$$H' = 1 + h_0 z^{1-\alpha_0} + h_1 z^{1+\alpha_0} + o\left(z^{2(1-\alpha_0)}\right) \tag{11}$$
$$\alpha = \alpha_0 - h_0 z^{1-\alpha_0} + h_1 z^{1+\alpha_0} + o\left(z^{2(1-\alpha_0)}\right)$$

where h_0 and h_1 are arbitrary constants. The corresponding secondary solution has the form

$$F = f_0 z^{1/2(1-\alpha_0)}\left[1 + o\left(z^{(1-\alpha_0)}\right)\right], \qquad G = g_0 z^{1/2(1+\alpha_0)}\left[1 + o\left(z^{(1-\alpha_0)}\right)\right] \tag{12}$$

The stagnation streamline near the stagnation point is described by

$$-\frac{x}{f_0} \doteq \frac{z^{1/2(1-\alpha_0)}}{2(1-\alpha_0)}, \qquad -\frac{y}{g_0} \doteq \frac{z^{1/2(1+\alpha_0)}}{2(1+\alpha_0)} \tag{13}$$

This result reduces to (10) if α_0 is set equal to zero.

P l a n a r C a s e $\alpha_0 = 1$. In this special case we require that $\alpha = H'$. We obtain for the primary solution

$$H = k^{-1}\sinh kz, \qquad \alpha = H' = \cosh kz \tag{14}$$

where k is arbitrary. In this case M cannot be set equal to zero in general. The secondary solution is

$$F = M \ln \frac{\sinh kz}{1 + \cosh kz} + f_0, \tag{15}$$

$$G = g_0 H \tag{16}$$

Unless both M and f_0 are zero no stagnation point can exist. Provided $M = 0$ and $f_0 = 0$, the stagnation streamlines near the stagnation points are described by

$$x = \text{const}, \qquad y = {}^1/_4 g_0 z \tag{17}$$

Here the streamlines come in to the wall at a finite angle.

A l m o s t - p l a n a r c a s e $\alpha_0 = 1$. In this case we exclude the possibility that $\alpha = H'$. A new variable ξ is defined by

$$\xi = \ln \frac{z_0}{z} \qquad \text{for} \qquad z = z_0 e^{-\xi} \tag{18}$$

where z_0 is an arbitrary constant. The primary solution has the form

$$\begin{aligned} H &= z + z\xi^{-1}\left(1 + o\left(\xi^{-1}\ln\xi\right)\right) \\ {}^1/_2\left(H' - \alpha\right) &= \xi^{-1}\left(1 + o\left(\xi^{-1}\ln\xi\right)\right) \\ {}^1/_2\left(H' + \alpha\right) &= 1 + {}^1/_2 k^2 z^2 \xi^2\left(1 + o\left(\xi^{-1}\ln\xi\right)\right) \end{aligned} \tag{19}$$

where k is an arbitrary constant.

The secondary solution has the form

$$F = -{}^1/_2 M\xi\left(1 + o\left(\xi^{-1}\ln\xi\right)\right) + {}^1/_2 f_0\left(H' - \alpha\right) \tag{20}$$

$$G = g_0\, z\xi\left(1 + o\left(\xi^{-1}\ln\xi\right)\right) \tag{21}$$

As in the planar case, M must be equal to zero in order that a stagnation point may exist. This means there can be no pressure gradient along the x-axis. If $f_0 \neq 0$ a translation of the origin in the x direction can be found which will make $f_0 = 0$. Thus we may set $f_0 = 0$ without loss of generality.

If $M = 0$ and $f_0 = 0$, a stagnation streamline enters the origin, with its shape described by $x = 0$ and

$$y = -\, {}^1/_4 g_0 z\,(\xi + {}^1/_2) \tag{22}$$

In this case, as in the planar case, not only is the origin a stagnation point but also all points along the x-axis. However, these other stagnation points have the somewhat unusual property that there exists no stagnation streamline entering any of them.

Saddle-point case $\alpha_0 > 1$. In this case the primary streamlines on the wall form a saddle-point pattern. The primary solution (11) of the nodal case applies here except that we must set $h_0 = 0$.

We obtain

$$H' - 1 = \alpha - \alpha_0 = h_1 z^{1+\alpha_0} + o\,(z^{2(1+\alpha_0)}) \tag{23}$$

with only one arbitrary constant. The secondary solution has the form

$$F = f_0 z^{1/2(1-\alpha_0)}\,[1 + o\,(z^{1+\alpha_0})],\; G = g_0 z^{1/2(1+\alpha_0)}\,[1 + o\,(z^{1+\alpha_0})] \tag{24}$$

In this case it is necessary that $f_0 = 0$ in order that a stagnation point may exist. If $f_0 = 0$ the stagnation streamline is described by $x = 0$ and the second equation of (13).

In summary, stagnation points of asisymmetric and nodal types (with $0 < \alpha_0 < 1$) can exist in quite great generality. A small perturbation in the pressure field or incoming vorticity distribution will only slightly perturb the stagnation point location and low field. Stagnation points of the types with $\alpha_0 \geqslant 1$ can exist only under special conditions. A perturbation of the pressure field making M nonzero will destroy the stagnation point in the cases $\alpha_0 = 1$, as will a small superposed uniform velocity in the x direction in the planar case. In the saddle-point case, any incoming vorticity with nonzero y component will destroy the stagnation point.

In the saddle-point case (with $f_0 = 0$) and with $\alpha_0 \leqslant 1$ if $f_0 = 0$ and $g_0 = 0$, the stagnation streamline comes in to the stagnation point normal to the wall. In the planar case with $g_0 \neq 0$ it comes in at a finite angle. In all the other cases considered the stagnation streamline comes in to the stagnation point tangent to the wall. This may be considered to be the situation in general in inviscid rotational flow.